Irreducibility Statistics

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Analogies between $\mathbf{Z}$ and $\mathbf{F}_p[u]$ for prime p

$\mathbf{Z}$	$\mathbf{F}_p[u]$
units are ± 1	units are $\mathbf{F}_p^\times$
prime	irreducible
composite	reducible
positive	monic
$\lvert m \rvert$	$\deg g$
Div. Algorithm	Div. Algorithm
$\mathbf{Z}/(m)$	$\mathbf{F}_p[u]/(g)$

Theorem. *For* $m \neq 0$, $|\mathbf{Z}/(m)| = |m|$; *for* $g \neq 0$, $|\mathbf{F}_p[u]/(g)| = p^{\deg g}$.

Ex. Why $|\mathbf{Z}/(9)| = 9$ and $|\mathbf{F}_3[u]/(u^2+1)| = 9$: in $\mathbf{Z}$, $a = 9q + r$ where $0 \leq r \leq 8$ so $a \equiv r \bmod 9$. In $\mathbf{F}_3[u]$, $g = (u^2+1)q + r$ where $r = c_1 u + c_0$ $(c_0, c_1 \in \mathbf{F}_3)$, so $g \equiv r \bmod u^2 + 1$.

Theorem. *For* $m \geq 1$, $\mathbf{Z}/(m)$ *is a field if and only if* m *is prime. For any* g, $\mathbf{F}_p[u]/(g)$ *is a field if and only if* g *is irreducible.*

Theorem. *For every prime* p, $\mathbf{F}_p^\times$ *is cyclic. For every irreducible* π, $(\mathbf{F}_p[u]/(\pi))^\times$ *is cyclic.*

For $g \neq 0$ in $\mathbf{F}_p[u]$ set $\mathrm{N}(g) = p^{\deg g} = |\mathbf{F}_p[u]/(g)|$. That is,

$$\mathrm{N}(c_d u^d + \cdots + c_1 u + c_0) = p^d$$

if $c_d \neq 0$.

Example. In $\mathbf{F}_3[u]$, $\mathrm{N}(2u^4 + u^2 + 2u + 2) = 3^4 = 81$.

Quadratic reciprocity works in $\mathbf{F}_p[u]$ when $p \neq 2$: for distinct *monic* irreducibles π_1 and π_2 in $\mathbf{F}_p[u]$,

$$\left(\frac{\pi_1}{\pi_2}\right)\left(\frac{\pi_2}{\pi_1}\right) = (-1)^{(\mathrm{N}(\pi_1)-1)/2 \cdot (\mathrm{N}(\pi_2)-1)/2}.$$

(There is a version of this when $p = 2$; more complicated.)

Nonanalogies between Z and $\mathbf{F}_p[u]$

1. Positive integers are closed under addition and multiplication. Monic polynomials are closed under multiplication but not addition.

2. Polynomials in $\mathbf{F}_p[u]$ can be composed: $g(h(u))$. Integers can't be composed.

3. Standard representatives of $\mathbf{F}_p[u]/(g)$ are closed under addition; this is not true in $\mathbf{Z}/(m)$. (The integers are *not* $(\mathbf{Z}/(10))[10]!$)

4. For prime powers p^k and $k > 1$, $(\mathbf{Z}/(p^k))^\times$ is usually cyclic: fails only if $p = 2$ and $k \geq 3$. For irreducible powers π^k with $k > 1$, $(\mathbf{F}_p[u]/(\pi^k))^\times$ is usually *not* cyclic: fails only if $\deg \pi = 1$ and $k = 2$, or if $\deg \pi = 1$, $p = 2$, and $k = 3$.

5. We can't prove $n > 1$ in $\mathbf{Z}$ is squarefree without factoring it, but $g \in \mathbf{F}_p[u]$ is squarefree (no repeated irred. factor) if and only if $(g(u), g'(u)) = 1$: no need to factor g.

Counting in $\mathbf{Z}$ and $\mathbf{F}_p[u]$

In $\mathbf{F}_p[u]$, count the size of the following sets and decide which should be the analogue of $\{1 \leq n \leq x\}$.

1. $\{g : \deg g = d\}$
2. $\{\text{monic } g : \deg g = d\}$
3. $\{g : \deg g \leq d\}$
4. $\{\text{monic } g : \deg g \leq d\}$

Analogue of Prime Number Theorem in $\mathbf{F}_p[u]$

Fix a prime number p. For $n \geq 1$, set

$$\mathrm{I}_p(n) = |\{\text{monic irreducibles in } \mathbf{F}_p[u] \text{ of degree } n\}|,$$

$$\mathrm{I}_p^{\leq}(n) = |\{\text{monic irreducibles in } \mathbf{F}_p[u] \text{ of degree up to } n\}|.$$

Then $\mathrm{I}_p(n) \sim \dfrac{p^n}{n}$ and $\mathrm{I}_p^{\leq}(n) \sim \dfrac{p^n}{n}\left(\dfrac{p}{p-1}\right)$ as $n \to \infty$.

n	5	6	7	8	9	10
$\mathrm{I}_3(n)$	48	116	312	810	2184	5580
$3^n/n$	48.6	121.5	312.42	820.12	2187.0	5904.9
Ratio	.9876	.9547	.9986	.9876	.9986	.9449
$\mathrm{I}_3^{\leq}(n)$	80	196	508	1318	3502	9382
$3^{n+1}/2n$	72.9	182.2	468.6	1230.1	3280.5	8857.3
Ratio	1.0973	1.0754	1.0839	1.0713	1.0675	1.0592

If we count nonmonic polynomials then multiply counts by $p-1$:
$\mathrm{I}_p^{\text{all}}(n) \sim (p-1)p^n/n$ and $\mathrm{I}_p^{\text{all},\leq}(n) \sim p^{n+1}/n$ as $n \to \infty$.

Analogue of Bunyakovsky's conjecture in $\mathbf{F}_p[u][T]$

Fix $f(T)$ in $\mathbf{F}_p[u][T]$ with $\deg_T(f) > 0$. Examples include

$$T^2 + u, \quad uT + u^4 - 1, \quad (u+1)T^3 + T + u^2.$$

Question: Do infinitely many $g \in \mathbf{F}_p[u]$ make $f(g)$ irreducible?

Example. Consider $f(T) = T^2 + u$ in $\mathbf{F}_3[u][T]$.

g	$f(g)$	Irreducible?
0	u	✓
1	$u+1$	✓
2	$u+1$	✓
u	u^2+u	
$u+1$	u^2+1	✓
$u+2$	u^2+2u+1	
$2u$	u^2+u	
$2u+1$	u^2+2u+1	
$2u+2$	u^2+1	✓
u^2+2	u^4+u^2+u+1	✓

Analogue of Bunyakovsky's conjecture in $\mathbf{F}_p[u][T]$

Two constraints on $f(T) \in \mathbf{F}_p[u][T]$ with $\deg_T f > 0$ in order for $f(g)$ to be irreducible in $\mathbf{F}_p[u]$ for infinitely many $g \in \mathbf{F}_p[u]$:

1. $f(T)$ is irreducible in $\mathbf{F}_p[u][T]$ (*i.e.*, $f \neq f_1 f_2$ for $f_1, f_2 \notin \mathbf{F}_p^\times$).
2. (Bunyakovsky condition) For each irreducible π in $\mathbf{F}_p[u]$ there is $g \in \mathbf{F}_p[u]$ such that $f(g) \not\equiv 0 \bmod \pi$.

Example. If $f(T) = T^n + u$ and $n \geq 1$ then the Bunyakovsky condition is satisfied because $f(0) = u$ and $f(1) = 1 + u$ are relatively prime. It is irreducible in $\mathbf{F}_p[u][T]$.

Example. If $f(T) = T^p - T + u$ then $f(g) \equiv 0 \bmod u$ for all $g \in \mathbf{F}_p[u]$, so $f(T)$ fails the Bunyakovsky condition at u. It is irreducible in $\mathbf{F}_p[u][T]$.

Conjecture. *If $f(T) \in \mathbf{F}_p[u][T]$ is nonconstant in T and fits both conditions above then there are infinitely many g in $\mathbf{F}_p[u]$ making $f(g)$ irreducible.*

Example of analogue of Bunyakovsky's conjecture in $\mathbf{F}_3[u][T]$

Set $f(T) = T^2 + u$ in $\mathbf{F}_3[u][T]$. As g runs over all polynomials of degree n in $\mathbf{F}_3[u]$, how often is $f(g) = g^2 + u$ irreducible?

n	Irred. values
4	16
5	46
6	104
7	256
8	694
9	1820
10	5028
11	13634
12	37674
13	104438
14	291610
15	815210
16	2291794

Conjecture. *If $f(T) \in \mathbf{F}_p[u][T]$ is nonconstant in T, irreducible in $\mathbf{F}_p[u][T]$, and fits the Bunyakovsky condition then $f(g)$ is irreducible for infinitely many g in $\mathbf{F}_p[u][T]$.*

The only proved case of this conjecture is when $\deg_T f = 1$, which says each $a + mT$ with $\gcd(a, m) = 1$ in $\mathbf{F}_p[u]$ has infinitely many irreducible values on $\mathbf{F}_p[u]$. This is an analogue of Dirichlet's theorem.

Conjecture. *If $f_1, \ldots, f_r \in \mathbf{F}_p[u][T]$ are nonconstant in T, irred. in $\mathbf{F}_p[u][T]$, and their product fits the Bunyakovsky condition then $f_1(g), \ldots, f_r(g)$ are irreducible for infinitely many g in $\mathbf{F}_p[u][T]$.*

Example. If $p > 2$ then $f_1(T) = T$ and $f_2(T) = T + c$ satisfy Hypothesis H in $\mathbf{F}_p[u]$ for $c \in \mathbf{F}_p^\times$, so we expect infinitely many irreducible pairs $\pi, \pi + c$ as π runs over irreducibles in $\mathbf{F}_p[u]$. In fact this is a theorem of Chris Hall (2006) if $p > 3$ and Paul Pollack if $p = 3$ (2008).

Analogue of Bateman–Horn conjecture in $\mathbf{F}_p[u][T]$ for 1 polynomial

For nonconstant irreducible $f(T) \in \mathbf{Z}[T]$ satisfying the Bunyakovsky condition, the Bateman–Horn conjecture predicts

$$\pi_f(x) \sim \frac{1}{\deg f} \prod_p \left(\frac{1 - \omega_f(p)/p}{1 - 1/p} \right) \cdot \frac{x}{\log x}.$$

For irreducible $f(T)$ in $\mathbf{F}_p[u][T]$ with $\deg_T(f) > 0$ set

$$\pi_f(n) = |\{g \text{ of degree } n \text{ in } \mathbf{F}_p[u] : f(g) \text{ is irreducible}\}|.$$

As $n \to \infty$ the number of irreducibles in $\mathbf{F}_p[u]$ of degree n is $\sim (p-1)p^n/n$, so it's natural to conjecture that

$$\pi_f(n) \sim \frac{1}{\deg_T f} \prod_{\text{monic } \pi} \left(\frac{1 - \omega_f(\pi)/\operatorname{N}(\pi)}{1 - 1/\operatorname{N}(\pi)} \right) \cdot \frac{(p-1)p^n}{n},$$

where $\omega_f(\pi) = |\{g \bmod \pi : f(g) \equiv 0 \bmod \pi\}|$ and $\operatorname{N}(\pi) = p^{\deg \pi}$.
Example. For $f(T) = T^2 + u$ in $\mathbf{F}_3[u][T]$, the conjecture says

$$\pi_f(n) \sim \frac{1}{2} \prod_{\pi \neq u} \left(1 - \frac{(-u|\pi)}{\operatorname{N}(\pi) - 1} \right) \cdot \frac{(3-1)3^n}{n}.$$

Example of analogue of Bateman–Horn conjecture in $\mathbf{F}_3[u][T]$

If $f = T^2 + u$, $C_f = \prod_{\pi \neq u} \left(1 - \frac{(-u|\pi)}{\mathrm{N}(\pi) - 1}\right)$ converges too slowly to compute to high accuracy directly, like $C = \prod_{p>2} \left(1 - \frac{(-1|p)}{p-1}\right)$.

n	7	8	9	10	11
Prod. for $\deg(\pi) \leq n$	.849564	.850852	.850850	.851195	.851195

Yesterday, to speed up convergence of C we rewrote it as

$$C = \frac{4}{\pi} \prod_{p \equiv 1 \bmod 4} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^2 - 1}\right)$$

and by similar reasoning we can rewrite C_f as

$$C_f = \prod_{\pi(0)=1} \left(1 - \frac{1}{(\mathrm{N}(\pi) - 1)^2}\right) \prod_{\pi(0)=2} \left(1 - \frac{1}{\mathrm{N}(\pi)^2 - 1}\right),$$

which converges more rapidly: $C_f \approx .8513304606$.

Example of analogue of Bateman–Horn conjecture in $\mathbf{F}_3[u][T]$

Count irreducible values of $f(T) = T^2 + u$ on $\mathbf{F}_3[u]$: as g runs over polynomials of degree n in $\mathbf{F}_3[u]$ we count how often $g^2 + u$ is irreducible. Is this number, $\pi_f(n)$, asymptotic to $\sim C_f(3^n/n)$ where $C_f \approx .8513304606$?

n	$\pi_f(n)$	$C_f(3^n/n)$	Ratio
7	256	265.9	.9624
8	694	698.1	.9939
9	1820	1861.8	.9775
10	5028	5027.0	1.0001
11	13634	13710.0	.9944
12	37674	37702.6	.9992
13	104438	104407.3	1.0002
14	291610	290849.0	1.0026
15	815210	814377.4	1.0010
16	2291794	2290436.5	1.0005

Example of analogue of Bateman–Horn conjecture in $\mathbf{F}_3[u][T]$

Count irreducible values of $f(T) = T^{12} + (u+1)T^6 + u^4$ on $\mathbf{F}_3[u]$: how often is $g^{12} + (u+1)g^6 + u^4$ irreducible as g runs over polynomials of degree n in $\mathbf{F}_3[u]$ as $n \to \infty$?

n	$\pi_f(n)$	Estimate	Ratio
9	1624	1168.3	1.390
10	4228	3154.5	1.340
11	11248	8603.2	1.307
12	31202	23658.7	1.319
13	87114	65516.5	1.330
14	244246	182510.2	1.338
15	683408	511028.6	1.337
16	1914254	1437268.0	1.332

The ratio isn't tending to 1! What will happen in the long run?

Example of analogue of Bateman–Horn conjecture in $\mathbf{F}_3[u][T]$

Count irreducible values of $f(T) = T^3 + u$ on $\mathbf{F}_3[u]$: how often is $g^3 + u$ irreducible as g runs over polynomials of degree n in $\mathbf{F}_3[u]$ as $n \to \infty$?

n	$\pi_f(n)$	Estimate	Ratio
9	1404	1458.0	.963
10	7776	3936.6	1.975
11	10746	10736.2	1.001
12	0	29524.5	0
13	82140	81760.2	1.005
14	455256	227760.4	1.999
15	637440	637729.2	1.000
16	0	1793613.4	0

Odd degrees look okay, but even degrees are surprising.

Example of analogue of Bateman–Horn conjecture in $\mathbf{F}_5[u][T]$

Count irreducible values of $f(T) = T^{10} + u$ on $\mathbf{F}_5[u]$: how often is $g^{10} + u$ irreducible as g runs over polynomials of degree n in $\mathbf{F}_5[u]$ as $n \to \infty$?

n	$\pi_f(n)$	Estimate	Ratio
4	0	125.0	0
5	0	500.0	0
6	0	2083.3	0
7	0	8928.6	0
8	0	12686.5	0
9	0	173611.1	0
10	0	781250.0	0
11	0	3551136.4	0
12	0	16276041.7	0
13	0	75120192.3	0
14	0	348772321.4	0
15	0	1627604166.7	0
16	0	7629394531.3	0

What's happening?

In $\mathbf{F}_5[u]$ let's see how $g^{10} + u$ factors for some nonconstant g.

$$
\begin{aligned}
u^{10} + u &= \pi_1 \widetilde{\pi}_1 \pi_2 \pi_6 \\
(u+1)^{10} + u &= \pi_1 \pi_9 \\
(u+2)^{10} + u &= \pi_1 \widetilde{\pi}_1 \pi_3 \pi_5 \\
(u+3)^{10} + u &= \pi_3 \pi_7 \\
(u+4)^{10} + u &= \pi_2 \pi_8 \\
(u^2)^{10} + u &= \pi_1 \widetilde{\pi}_1 \pi_9 \widetilde{\pi_9} \\
(u^2+1)^{10} + u &= \pi_1 \pi_2 \widetilde{\pi}_2 \pi_{15} \\
(u^2+2)^{10} + u &= \pi_1 \pi_3 \pi_6 \pi_{10} \\
(u^2+3)^{10} + u &= \pi_1 \pi_{19} \\
(u^2+4)^{10} + u &= \pi_3 \pi_{17} \\
(u^2+u+2)^{10} + u &= \pi_2 \pi_4 \pi_7 \widetilde{\pi}_7 \\
(u^3+2)^{10} + u &= \pi_1 \widetilde{\pi}_1 \pi_4 \pi_6 \pi_7 \pi_{11}
\end{aligned}
$$

What patterns do you see in the irreducible factorizations?

What's happening?

Theorem. *For nonconstant g in $\mathbf{F}_5[u]$, $g^{10} + u$ has an even number of irreducible factors.*

To prove this we use an analogue of a number-theoretic function more obscure than $\varphi(n)$ and $\sigma(n)$: the Möbius function $\mu(n)$.

Definition. Set $\mu(1) = 1$, and for $n > 1$ set

$$\mu(n) = \begin{cases} (-1)^r, & \text{if } n = p_1 \cdots p_r \text{ (squarefree)}, \\ 0, & \text{otherwise.} \end{cases}$$

Example. Compute $\mu(2)$, $\mu(4)$, $\mu(15)$, and $\mu(45)$.

There is no known way to compute $\mu(n)$ without factoring n.

Definition. Set $\mu(c) = 1$ for $c \in \mathbf{F}_p^\times$. For nonconstant $g \in \mathbf{F}_p[u]$,

$$\mu(g) = \begin{cases} (-1)^r, & \text{if } g = \pi_1 \cdots \pi_r \text{ (squarefree)}, \\ 0, & \text{otherwise.} \end{cases}$$

For irreducible π, $\mu(\pi) = -1$. Thus $\mu(g) = 1 \Rightarrow g$ isn't irreducible.

What's happening?

Unlike in $\mathbf{Z}$, in $\mathbf{F}_p[u]$ there is a formula for $\mu(g)$ other than its definition! Using properties of discriminants and resultants, if $p \neq 2$ and g is squarefree in $\mathbf{F}_p[u]$ with r (monic) irreducible factors then

$$\left(\frac{\text{disc } g}{p}\right) = (-1)^{\deg g - r},$$

so

$$\mu(g) = (-1)^r = (-1)^{\deg g} \left(\frac{\text{disc } g}{p}\right).$$

This last formula for $\mu(g)$ is also valid for g not squarefree since disc $g = 0$. Using properties of discriminants and resultants on the special problem set, $\mu(g)$ can be computed without factoring g. This refines being able to prove g is squarefree ($\mu(g) \neq 0$) without factoring g by checking that $(g, g') = 1$.

Theorem. *For nonconstant g in $\mathbf{F}_5[u]$, $\mu(g^{10} + u) = 1$ so $g^{10} + u$ is not irreducible.*

A Möbius bias on polynomial sequences in $\mathbf{F}_p[u]$

All known $f(T)$ violating the expected analogue of the Bateman–Horn conjecture in $\mathbf{F}_p[u][T]$ have three properties.

1. They are polynomials in T^p, *e.g.*, $T^{10} + u = (T^5)^2 + u$.
2. The bad ratio statistics seem to have 1, 2, or 4 limiting values.
3. A Möbius bias: averages of $\mu(f(g))$ stay **away** from 0.

Example. If $f(T) = T^{12} + (u+1)T^6 + u^4$ in $\mathbf{F}_3[u][T]$ then

$$n \geq 2 \Longrightarrow \frac{\sum_{\deg g = n} \mu(g^{12} + (u+1)g^6 + u^4)}{\sum_{\deg g = n} |\mu(g^{12} + (u+1)g^6 + u^4)|} = -\frac{1}{3}.$$

Example. If $f(T) = T^3 + u$ in $\mathbf{F}_3[u][T]$ then

$$n \geq 1 \Longrightarrow \frac{\sum_{\deg g = n} \mu(g^3 + u)}{\sum_{\deg g = n} |\mu(g^3 + u)|} = 0, -1, 0, 1, 0, -1, 0, 1, \ldots.$$

Example. If $f(T) = T^{10} + u$ in $\mathbf{F}_5[u][T]$ then

$$n \geq 1 \Longrightarrow \frac{\sum_{\deg g = n} \mu(g^{10} + u)}{\sum_{\deg g = n} |\mu(g^{10} + u)|} = 1.$$

No classical Möbius bias

The Prime Number Theorem is *equivalent* to

$$\frac{1}{x}\sum_{n\le x}\mu(n)\to 0 \text{ as } x\to\infty.$$

It is believed that for irreducible $f(T)\in\mathbf{Z}[T]$ satisfying the Bunyakovsky condition that

$$\frac{1}{x}\sum_{n\le x}\mu(f(n))\to 0 \text{ as } x\to\infty,$$

but it is only proved for linear $f(T)$ (PNT and Dirichlet's theorem).

Can you tell which of the following sequences of twenty successive nonzero Möbius values is taken from somewhere in the sequence $\mu(n)$ or the sequence $\mu(n^2+1)$?

$$1,1,-1,-1,-1,-1,-1,1,1,1,1,1,-1,1,1,-1,-1,1,1,-1$$

$$-1,1,1,1,-1,1,1,1,1,-1,-1,-1,-1,-1,1,-1,-1,-1,1,-1$$

Theorem. (KC, BC, R. Gross) *Let* $p \neq 2$ *and* $f(T) \in \mathbf{F}_p[u][T]$ *be nonconstant in* T*, irreducible, and fit the Bunyakovsky condition. If* $f(T) \in \mathbf{F}_p[u][T^p]$ *then the sequence*

$$\Lambda_p(f;n) := 1 - \frac{\sum_{\deg g=n} \mu(f(g))}{\sum_{\deg g=n} |\mu(f(g))|}$$

for sufficiently large n *is periodic in* n *with period* 1, 2, *or* 4.

The sequence $\Lambda_p(f;n)$ is much simpler than its definition suggests.

p	$f(T)$	Bad ratio	$\Lambda_p(f;n)$
3	$T^{12} + (u+1)T^6 + u^4$	1.33	$4/3$
3	$T^3 + u$	1.00, 1.99, 1.00, 0	1, 2, 1, 0
5	$T^{10} + u$	0	0

Note. The correct definition of $\Lambda_p(f;n)$ should have an additional (technical) condition in the averaging over g. In some, but not all, examples ignoring the condition doesn't affect $\Lambda_p(f;n)$ for large n.

Correction to analogue of Bateman–Horn conjecture in $\mathbf{F}_p[u][T]$

Conjecture. (KC, BC, R. Gross) *Let $p \neq 2$ and $f(T) \in \mathbf{F}_p[u][T]$ be nonconstant in T, irreducible, and satisfy the Bunyakovsky condition. If $f(T)$ is not a polynomial in T^p then as $n \to \infty$*

$$\pi_f(n) \sim \frac{1}{\deg_T f} \prod_{\text{monic } \pi} \left(\frac{1 - \omega_f(\pi)/\operatorname{N}(\pi)}{1 - 1/\operatorname{N}(\pi)} \right) \cdot \frac{(p-1)p^n}{n}.$$

If $f(T)$ is a polynomial in T^p then as $n \to \infty$

$$\pi_f(n) \sim \Lambda_p(f;n) \frac{1}{\deg_T f} \prod_{\text{monic } \pi} \left(\frac{1 - \omega_f(\pi)/\operatorname{N}(\pi)}{1 - 1/\operatorname{N}(\pi)} \right) \cdot \frac{(p-1)p^n}{n}.$$

Summary. In $\mathbf{Z}[T]$ all obstructions to an irreducible polynomial having infinitely many prime values are expected to be "local" (there is a problem mod p for some prime p). In $\mathbf{F}_p[u][T]$ there is an extra obstruction to taking irreducible values infinitely often that is that is "global" (unrelated to modular arithmetic) and is due to a bias in Möbius averages having no classical analogue.

What about $p = 2$?

- In $\mathbf{F}_2[u][T^2]$ there are polynomials that deviate numerically from the analogue of the Bateman–Horn conjecture.
- For $g \in \mathbf{F}_2[u]$ there is a formula for $\mu(g)$ other than its definition, but it's more complicated than in $\mathbf{F}_p[u]$ for $p \neq 2$.
- For $f(T) \in \mathbf{F}_2[u][T^4]$ the Möbius bias factor $\Lambda_2(f; n)$ fits the data and provably has period 1, 2, or 4, but polynomials in T^2 that are not polynomials in T^4 are still mysterious.

n	$\pi_{T^6+u}(n)$	Estimate	Ratio	$\Lambda_2(T^6 + u; n)$
11	0	43.0	0	0
12	0	78.8	0	0
13	306	145.5	2.1018	2
14	512	270.3	1.8937	2
15	0	504.6	0	0
16	0	946.2	0	0
17	3575	1781.2	2.0070	2
18	6726	3364.5	1.9990	2